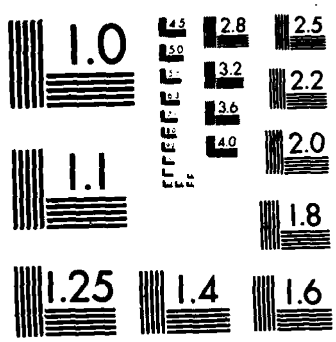
AD-A137 958 ON THE ASYMPTOTICS OF MAXIMUM LIKELIHOOD AND RELATED ESTIMATORS BASED ON..(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER G K BHATTACHARYYA DEC 83 1/1

UNCLASSIFIED MRC-TSR-2616 DAAG29-80-C-0041 F/G 12/1 NL

END
FILMED
3-84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A137958

MRC Technical Summary Report #2616

ON THE ASYMPTOTICS OF MAXIMUM LIKELIHOOD AND RELATED ESTIMATORS BASED ON TYPE II CENSORED DATA

Gouri K. Bhattacharyya

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

December 1983

(Received November 7, 1983)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Office of Naval Research
Department of the Navy
Arlington, VA 22217

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

ON THE ASYMPTOTICS OF MAXIMUM LIKELIHOOD AND RELATED ESTIMATORS BASED ON TYPE II CENSORED DATA

Gouri K. Bhattacharyya

Technical Summary Report #2616

December 1983

ABSTRACT

Some simple procedures are provided for establishing the asymptotic normality and uniform strong convergence of a class of functions that arise in the context of estimating parameters from a type II censored sample. These are used to streamline and strengthen the traditional treatment of the asymptotic theory of maximum likelihood estimators based on censored data. Further applications include the treatment of asymptotics of some modified maximum likelihood (MML) estimators. In particular, conditions are provided for the consistency and limiting normality of the MML estimators of Mehrotra and Nanda, and the asymptotic efficiencies of these estimators are evaluated.

AMS (MOS) Subject Classifications: Primary 62F12; Secondary 62N05

Key Words: Censored data; Maximum likelihood; Modified maximum likelihood; Asymptotic theory.

Work Unit Number 4 - Statistics and Probability

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
Supported by the Office of Naval Research Grant N00014-78-C-0722.

SIGNIFICANCE AND EXPLANATION

In reliability analysis, inferences on the parameters of a life distribution often have to be based on censored or incompletely observed data. Under the censoring scheme that permits observation of a predetermined number of failures, several modifications of the maximum likelihood method were proposed with the goal of obtaining estimators that are relatively easy to apply. Previous works on the large-sample properties of maximum likelihood and modified maximum likelihood estimators have been rather sketchy, and the methods too cumbersome to employ in multiple-censoring situations.

This paper develops a simple yet versatile approach that permits a unified treatment of the large-sample properties of both maximum likelihood and modified maximum likelihood estimators based on censored data. It is also flexible enough to accommodate multiple censoring. In addition to providing an improved theoretical treatment, the results help fill a gap of knowledge in regard to the performance of the modified maximum likelihood estimators, especially their loss of efficiency in relation to the severity of censoring.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE ASYMPTOTICS OF MAXIMUM LIKELIHOOD AND RELATED ESTIMATORS BASED ON TYPE II CENSORED DATA

Gouri K. Bhattacharyya

1. INTRODUCTION

In a life-test setting, suppose the failure times of n units be modeled as independent random variables $X_1,\ldots,X_n$ having a common continuous distribution. In general term, a type II censored sample refers to a specified subset of the order statistics $Y_1 < \ldots < Y_n$ of $X_1,\ldots,X_n$. The most common situation is censoring on the right which permits the first r ($< n$) order statistics to be observed. However, left as well as multiple censoring are also often used.

Letting $f(x,\theta)$ and $F(x,\theta)$ denote the probability density function (pdf) and the distribution function (df) of the failure time, the log-likelihood of a type II right censored sample is

$$\ell_n(\theta) \equiv \log[n!/(n-r)!] + \sum_{i=1}^{r} \log f(Y_i,\theta) + (n-r)\log \bar{F}(Y_r,\theta) \qquad (1.1)$$

where $\bar{F} = 1-F$. Maximum likelihood (ML) estimation under specific parametric models of the life distribution is widely discussed in the reliability literature. In regard to the asymptotic theory of ML, the standard theorems do not apply to (1.1) because its terms are neither independent nor identically distributed. The work of Halperin (1952) still remains the universal reference for a rigorous treatment of the asymptotics of ML in this context. However, Halperin's approach, which rests on asymptotic expansions of certain characteristic functions, involves quite tedious manipulations even for the simplest case where θ is real and the sample is single-censored.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.
Supported by the Office of Naval Research Grant N00014-78-C-0722.

A fair amount of work has grown on another front. In order to reduce the computing job of itertively solving the likelihood equation or to obtain simple estimators that permit a grip on their small-sample properties, some modified maximum likelihood (MML) estimators have been proposed. These are generally targeted for location-scale models, and especially, for the normal distribution. One interesting construct is due to Mehrotra and Nanda (1974) who replace the 'hazard-rate' term $\partial \log \bar{F}(Y_r,\theta)/\partial\theta$, that appears in the likelihood equation, by its expectation. While the small sample properties of these MML estimators have been studied for some particular models, their asymptotic efficiencies have not been investigated. Another type of MML estimators, due to Tiku (1967, 1978), derives from a linear approximation of the hazard-rate term. Although extensive simulation studies of these estimators have been reported, a careful treatment of the asymptotic theory is lacking.

The objectives of the present paper are twofold. First, we streamline and strengthen the traditional treatment of the asymptotics of ML estimators with type II censored data. At the same time, we provide a general setting in which the asymptotics of ML, the aforementioned MML's or other perturbations of the estimating equation can be treated in a unified way. Our second goal is to derive the asymptotic efficiency (AE) of the Mehrotra and Nanda MML estimator and study the effect of the amount of censoring on the AE.

The key results about limiting normality and uniform strong convergence are developed in Section 2. These are appropriately specialized in Sections 3 and 4 to handle the ML and MML estimators. Our treatment of asymptotic normality is based on a result of Sethuraman (1961) concerning the conditional and joint limit distribution of random vectors. It yields a considerable simplification over Halperin's treatment, almost to the level of simplicity of the iid case. Moreover, it makes the adaptation of the results to the multi-censoring case quite transparent. Incidentally, Fligner and Hettmansperger (1979) made another fruitful use of this approach in the context of some rank statistics.

Results for the Mehrotra and Nanda MML estimator are developed in Section 4 in a general setting, and then specialized to the location and scale parameters in Section 5. Numerical computations of the asymptotic variance and AE are presented in Section 5 under the normal model for which these estimators were found to have nice small-sample properties. These supplement the small-sample variance and efficiency calculations of Mehrotra and Nanda (1974), and show the effect of the amount of censoring. Section 6 briefly indicates how the asymptotics of Tiku's MML estimator follow from our results.

2. THE PRINCIPAL TOOLS

This section provides two results which are basic to the development of asymptotic theory of ML and MML estimators based on type II censored data. First we introduce some notation and assumptions. The parameter θ is taken to be a k-vector and the true value θ_o is assumed to be an interior point of the parameter space $\Omega \subset R^k$. An important role will be played by random vectors of the form

$$T_n(\theta) = n^{-1}[\sum_{i=1}^{r} g(Y_i,\theta) + (n-r)h(Y_r,\theta)] \tag{2.1}$$

where g and h are functions on $\mathcal{X} \times \Omega \to R^{\ell}$, and $\mathcal{X}$ denotes the sample space of X_1. For simplicity, θ_o will often be suppressed in functional notation. For instance, $f(x)$ will stand for $f(x,\theta_o)$, $g(x)$ for $g(x,\theta_o)$, and

$$T_n = T_n(\theta_o) = n^{-1}[\sum_{i=1}^{r} g(Y_i) + (n-r)h(Y_r)] \quad . \tag{2.2}$$

Unless specified otherwise, all limits are taken as $n \to \infty$, and r is taken to be $[np]$, the integer part of np where $p \in (0,1)$ is fixed. Throughout it is assumed that $f(x)$ is continuous at its p-quantile ζ, and $f(\zeta) > 0$. The notation $\overset{\mathcal{D}}{\to} N_k(\mu,\Sigma)$ will be used for convergence in distribution to a k-variate normal with mean μ and covariance matrix Σ. Vectors will be written as column vectors and a transpose will be denoted by *.

Our first concern is with the asymptotic distribution of T_n defined in (2.2). Instead of working with sophisticated limit theorems for functions of order statistics, we provide an elementary treatment by means of conditioning on Y_r. A result of Sethuraman (1961), stated in Lemma 1, would be instrumental to our approach.

<u>Lemma 1</u>. Let $\{\xi_n\}$ and $\{\eta_n\}$ be sequences of random ℓ- and m-vectors defined on a probability space. If (a) for arbitrary $t \in R^m$, the conditional distribution of ξ_n, given $\eta_n = t$, converges to $N_\ell(Bt,\tau)$, and (b) η_n strongly converges in distribution to $N_m(0,\Delta)$, then $\xi_n \overset{\mathcal{D}}{\to} N_\ell(0,\tau + B\Delta B^*)$.

Referring to (2.2), let g_α and h_α denote the αth coordinate of g and h, $\alpha = 1,\ldots,\ell$. Theorem 1 establishes the asymptotic normality of T_n.

<u>Theorem 1</u>. For $\alpha = 1,\ldots,\ell$, assume that (i) $h'_\alpha(x) \equiv dh_\alpha(x)/dx$ exists at $x = \zeta$, (ii) $g_\alpha(x)$ is continuous at ζ, and (iii) $\int_{-\infty}^{\zeta} g_\alpha^2(x)f(x)dx < \infty$. Then $n^{1/2}(T_n-\mu) \overset{\mathcal{D}}{\to} N_\ell(0,\Sigma)$ where

$$
\begin{aligned}
\mu &= \int_{-\infty}^{\zeta} g(x)f(x)dx + qh(\zeta) \\
\Sigma &= \tau + pqf^{-2}(\zeta)bb^* \\
\tau &= \int_{-\infty}^{\zeta} g(x)g^*(x)f(x)dx - p^{-1}\left(\int_{-\infty}^{\zeta} g(x)f(x)dx\right)\left(\int_{-\infty}^{\zeta} g(x)f(x)dx\right)^* \\
b &= f(\zeta)g(\zeta) - p^{-1}f(\zeta)\int_{-\infty}^{\zeta} g(x)f(x)dx + qh'(\zeta) \quad .
\end{aligned}
\tag{2.3}
$$

Proof. Observe that a linear function c^*T_n, of the components of T_n, is again a function of the form (2.2) with $\ell = 1$, and c^*g and c^*h in places of g and h. Therefore, it suffices to prove the result for the one-dimensional case ($\ell = 1$). We take g, h, and T_n to be real-valued functions for the rest of the proof.

Letting $\eta_n = n^{1/2}(Y_r - \zeta)$, consider first the conditional limit distribution of $n^{1/2}(T_n - \mu)$. Given $\eta_n = t$ or equivalently $Y_r = \zeta_n \equiv \zeta + tn^{-1/2}$, the random variables $Y_1, \ldots, Y_{r-1}$ are distributed as the order statistics of a random sample of size $r - 1$ from the truncated pdf $f(x)/F(\zeta_n)$, $x < \zeta_n$. Let

$F(\zeta_n) = p_n$ and denote the truncated moments of $g(x)$ as

$$\nu_{1n} = p_n^{-1} \int_{-\infty}^{\zeta_n} gf dx$$

$$\nu_{2n} = p_n^{-1} \int_{-\infty}^{\zeta_n} g^2 f dx - \nu_{1n}^2 \quad .$$

We have the decomposition $n^{1/2}(T_n - \mu) = A_{1n} + A_{2n}$ where

$$A_{1n} = n^{-1/2} \sum_{i=1}^{r-1} [g(Y_i) - \nu_{1n}]$$

$$A_{2n} = n^{1/2} \left[\frac{(r-1)}{n} \nu_{1n} + \frac{(n-r)}{n} h(\zeta_n) - \mu\right] + n^{-1/2} g(\zeta_n) \quad .$$

Conditionally, given $\eta_n = t$, A_{1n} is a sum of iid components centered at the mean. We are in fact dealing with a triangular array since the common distribution depends on n. By virtue of the assumptions (ii) and (iii), the Lindeberg-Feller central limit theorem applies, and we have that $A_{1n} \xrightarrow{\mathcal{D}} N_1(0,\tau)$ where

$$\tau = p(\lim \nu_{2n}) = \int_{-\infty}^{\zeta} g^2 f dx - p^{-1}\left(\int_{-\infty}^{\zeta} gf dx\right)^2 \quad .$$

The constant A_{2n} can be written as

$$A_{2n} = n^{1/2}\left[\frac{(r-1)}{n} p_n^{-1} \int_{-\infty}^{\zeta_n} gf dx - \int_{-\infty}^{\zeta} gf dx\right]$$

$$+ n^{1/2}\left[\frac{(n-r)}{n} h(\zeta_n) - qh(\zeta)\right] + n^{-1/2} g(\zeta_n) \quad .$$

Using assumptions (i) and (ii) along with the relation $n^{1/2}(\zeta_n - \zeta) = t$, it is straightforward to see that the three terms on the rhs converge to $tf(z)[g(\zeta) - p^{-1}\int_{-\infty}^{\zeta} gf dx]$, $tqh'(\zeta)$ and 0 respectively. Consequently, the conditional limit distribution of $n^{1/2}(T_n - \mu)$ is $N_1(bt, \tau)$ where

$$b = f(\zeta)g(\zeta) - p^{-1} f(\zeta) \int_{-\infty}^{\zeta} gf dx + qh'(\zeta) \quad .$$

Finally, η_n converges in density to $N_1(0, pqf^{-2}(\zeta))$ which implies strong convergence of η_n in distribution in the sense of Sethuraman (1961). An application of Lemma 1 yields that $n^{1/2}(T_n - \mu)$ is asymptotically normal with mean 0 and variance $\tau + pqf^{-2}(\zeta)b^2$. ||

<u>Remark</u>. The above approach to proving Theorem 1 readily extends to the case of multiple censoring. To illustrate, let us consider left censoring at the observation $r_1 = [np_1]$ and right censoring at $r_2 = [np_2]$, $0 < p_1 < p_2 < 1$. In this case, the relevant form of T_n is

$$n^{-1}\left[r_1 h_1(Y_{r_1}) + \sum_{i=r_1}^{r_2} g(Y_i) + (n-r_2) h_2(Y_{r_2})\right] \quad .$$

With ζ_α denoting the p_α-quantile of $f(x)$, $\alpha = 1,2$, the vector $\eta_n = n^{1/2}(Y_{r_1} - \zeta_1, Y_{r_2} - \zeta_2)^*$ converges in density to a bivariate normal $N_2(0, \Delta)$, say. Conditionally, given $\eta_n = t$, the middle term of T_n is distributed as a sum of iid components where the parent distribution has the doubly truncated pdf $f(x)/[F(\zeta_2) - F(\zeta_1)]$, $\zeta_1 < x < \zeta_2$. Following the steps of proof of Theorem 1, one arrives at the asymptotic normality of T_n along with an explicit expression for the asymptotic covariance matrix. ||

Our second result pertains to the convergence of $T_n(\theta)$, defined in (2.1), with probability 1 (w.p.1) uniformly in a compact neighborhood of θ_o. Since the components of a vector $T_n(\theta)$ can be individually treated for this purpose, it suffices to consider the case of real-valued functions g and h in Theorem 2.

<u>Theorem 2</u>. Assume that for a compact neighborhood B of θ_o,

(i) $g(x,\theta)$ is continuous in $\theta \in B$ for every x,

(ii) for $\theta \in B$ and all x, $|g(x,\theta)| \leq g_o(x)$ such that $\int_{-\infty}^{\infty} g_o(x)f(x)dx < \infty$,

(iii) $h(x,\theta)$ is continuous on $[\zeta-\varepsilon_1,\zeta+\varepsilon_1] \times B$ for some $\varepsilon_1 > 0$.

Then

$$\sup_{\theta \in B} |T_n(\theta) - \mu(\theta)| \to 0 \quad \text{w.p.1} \tag{2.4}$$

where

$$\mu(\theta) = \int_{-\infty}^{\zeta} g(x,\theta)f(x)dx + qh(\zeta,\theta) \quad . \tag{2.5}$$

<u>Proof</u>. Referring to (2.1), let $T_{1n}(\theta) = n^{-1}\sum_{i=1}^{r} g(Y_i,\theta)$ and $T_{2n}(\theta) = n^{-1}(n-r)h(Y_r,\theta)$ so $T_n(\theta) = T_{1n}(\theta) + T_{2n}(\theta)$. With $\chi_A(\cdot)$ denoting the indicator function of the set A, we have the representation

$$T_{1n}(\theta) = n^{-1}\sum_{i=1}^{n} g(X_i,\theta)\chi_{A(Y_r)}(X_i)$$

where $A(Y_r) = (-\infty, Y_r]$. Consider its approximation by $T_{1n}^{o}(\theta) \equiv n^{-1}\sum_{i=1}^{n} g(X_i,\theta)\chi_{A(\zeta)}(X_i)$ which is an average of iid components that are continuous in θ and bounded by the integrable g_o. The uniform strong law (cf. Jennrich 1969) yields

$$\sup_{\theta \in B} |T_{1n}^{o}(\theta) - \int_{-\infty}^{\zeta} g(x,\theta)f(x)dx| \to 0 \quad \text{w.p.1} \quad .$$

Now, for an arbitrary $\varepsilon > 0$, as $n \to \infty$, Y_r lies outside the interval $\zeta \pm \varepsilon$ at most finitely many times w.p.1. Consequently, for sufficiently large n,

$$|T_{1n}(\theta) - T^{o}_{1n}(\theta)| < n^{-1} \sum_{i=1}^{n} |g(X_i,\theta)| \chi_{[\zeta-\varepsilon,\zeta+\varepsilon]}(X_i)$$

$$< K\, n^{-1} \sum_{i=1}^{n} \chi_{[\zeta-\varepsilon,\zeta+\varepsilon]}(X_i)$$

where K bounds $g(x,\theta)$ on $[\zeta-\varepsilon,\zeta+\varepsilon] \times B$. The last expression converges to $K[F(\zeta+\varepsilon) - F(\zeta-\varepsilon)]$ w.p.1. By letting $\varepsilon \to 0$, we therefore obtain that $|T_{1n}(\theta) - T^{o}_{1n}(\theta)| \to 0$ w.p.1 uniformly in B. With regard to $T_{2n}(\theta)$, note that $\sup_{\theta \in B} |h(x,\theta) - h(\zeta,\theta)|$ is a continuous function $x \in [\zeta-\varepsilon_1,\zeta+\varepsilon_1]$. Since $Y_r \to \zeta$ w.p.1, we have that $|T_{2n}(\theta) - qh(\zeta,\theta)|$ 0 w.p.1 uniformly in $\theta \in B$. By combining the two parts, the proof is comp. ||

3. APPLICATION TO MLE

We proceed to show how the asymptotic normality and consistency results for the maximum likelihood estimator (MLE) in the type II censored situation can be obtained from simple adaptations of Theorems 1 and 2. It would be enough to outline the main steps because the details are analogous to the treatment of MLE in the iid case. Moreover, instead of displaying a composite list of regularity conditions, it would be more instructive to state them as and when they are needed.

Henceforth, we use an upper dot for the derivative of a function wrt θ, and two dots for the second derivative. Referring to (1.1) and denoting

$$\psi(x,\theta) = \log f(x,\theta),\ \rho(x,\theta) = \log \bar{F}(x,\theta) \quad , \qquad (3.1)$$

the likelihood equation is given by

$$\dot{\ell}_n(\theta) \equiv \sum_{i=1}^{r} \dot{\psi}(Y_i,\theta) + (n-r)\dot{\rho}(Y_r,\theta) = 0 \quad . \qquad (3.2)$$

Since θ is a k-vector, so is $\dot{\ell}_n(\theta)$ while $\ddot{\ell}_n(\theta)$ will be a $k \times k$ matrix. As before, the true θ_o will often be suppressed in notation, for instance, $\dot{\psi}(x) = \dot{\psi}(x,\theta_o)$.

First, we establish the asymptotic normality of $n^{1/2}(\hat{\theta}_n - \theta_o)$ assuming that $\{\hat{\theta}_n\}$ is a consistent sequence of roots. A Taylor expansion of $\dot{\ell}_n(\hat{\theta}_n) = 0$ around θ_o yields

$$n^{-1/2}\dot{\ell}_n(\theta_o) = \Gamma_n[n^{1/2}(\hat{\theta}_n - \theta_o)] \tag{3.3}$$

where Γ_n is the random matrix $-n^{-1}\ddot{\ell}_n(\cdot)$ evaluated on the line segment between $\hat{\theta}_n$ and θ_0. Now, $n^{-1}\dot{\ell}_n(\theta_o)$ is a vector-valued function of the form (2.2) with $\ell = k$, $g = \dot{\psi}$, and $h = \dot{\rho}$. Theorem 1, specialized to these g and h, readily yields the limiting normality. To obtain explicit expressions for the mean and covariance, we assume that the derivative $\frac{\partial}{\partial\theta}\int_{-\infty}^{x} f(y,\theta)dy$ can be carried within the integral. This yields

$$\begin{aligned} h(x) &= \dot{\rho}(x) = -[\bar{F}(x)]^{-1}\int_{-\infty}^{x} g(y)f(y)dy \quad , \\ h'(x) &= [\bar{F}(x)]^{-1}[-g(x)f(x) + h(x)] \quad . \end{aligned} \tag{3.4}$$

Using these results for $x = \zeta$, the expressions (2.3) for the present case reduce to

$$\begin{aligned} &\mu = 0,\ b = -(pq)^{-1}f(\zeta)\int_{-\infty}^{\zeta}\dot{\psi}f dx \\ &\Sigma \equiv J = \int_{-\infty}^{\zeta}\dot{\psi}\,\dot{\psi}^* f dx + q^{-1}\Big(\int_{-\infty}^{\zeta}\dot{\psi}f dx\Big)\Big(\int_{-\infty}^{\zeta}\dot{\psi}f dx\Big)^* \quad . \end{aligned} \tag{3.5}$$

To establish that $n^{1/2}(\hat{\theta}_n - \theta_o) \xrightarrow{\mathcal{D}} N_k(0,J^{-1})$, it remains to show that $\Gamma_n \to J$ w.p.1. To this end, we note that $n^{-1}\ddot{\ell}_n(\theta)$ is a (matrix-valued) function of the form $T_n(\theta)$ given in (2.1) with $g = \ddot{\psi}(x,\theta)$ and $h = \ddot{\rho}(x,\theta)$. Denoting

$$-J(\theta) \equiv \int_{-\infty}^{\zeta}\ddot{\psi}(x,\theta)f(x)dx + q\ddot{\rho}(\zeta,\theta) \quad ,$$

Theorem 2 entails that, uniformly in $\theta \in B$, $-n^{-1}\ddot{\ell}_n(\theta) \to J(\theta)$ w.p.1. Assuming $J(\theta)$ continuous at θ_o, it then follows that $\Gamma_n \to J(\theta_o)$ w.p.1. In order to relate $J(\theta_o)$ to J we differentiate $\int_{-\infty}^{\zeta}f(x,\theta)dx + \bar{F}(\zeta,\theta) = 1$

twice wrt θ and obtain the identity

$$- \int_{-\infty}^{\zeta} \ddot{\psi}(x,\theta)f(x,\theta)dx - \bar{F}(\zeta,\theta)\ddot{\beta}(\zeta,\theta)$$

$$= \int_{-\infty}^{\zeta} \dot{\psi}(x,\theta)\dot{\psi}^{*}(x,\theta)dx + [\bar{F}(\zeta,\theta)]^{-1}$$

$$(\int_{-\infty}^{\zeta} \dot{\psi}(x,\theta)f(x,\theta)dx)(\int_{-\infty}^{\zeta} \dot{\psi}(x,\theta)f(x,\theta)dx)^{*} \quad .$$

For $\theta = \theta_o$, the lhs equals $J(\theta_o)$ while the rhs equals J. This concludes the proof of the asymptotic normality of $n^{1/2}(\hat{\theta}_n - \theta_o)$.

Turning now to the issue of existence of a strongly consistent sequence of roots $\{\hat{\theta}_n\}$, we examine the limiting behavior of the log likelihood ratio $n^{-1}[\ell_n(\theta) - \ell_n(\theta_o)]$. This is again of the form $T_n(\theta)$ in (2.1), now a real-valued function, with the special g and h given by

$$g = \log[f(x,\theta)/f(x)], \; h = \log[\bar{F}(x,\theta)/\bar{F}(x)] \quad .$$

Theorem 2 entails that

$$n^{-1}[\ell_n(\theta) - \ell_n(\theta_o)] \to \mu(\theta) \equiv \int_{-\infty}^{\zeta} \log[f(x,\theta)/f(x)]f(x)dx + q \log[\bar{F}(\zeta,\theta)/q] \quad , \tag{3.6}$$

w.p.1 and uniformly in $\theta \in B$. Evidently $\mu(\theta_o) = 0$. To show that $\mu(\theta)$ has a local maximum at θ_o, we fix $\theta \in B$, define the function $u(x)$ as

$$\begin{aligned} u(x) &= f(x,\theta)/f(x) \quad , \quad x < \zeta \\ &= \bar{F}(\zeta,\theta)/q \quad , \quad x \geq \zeta \end{aligned}$$

and let Z be a random variable whose distribution has the pdf $f(x)$ on $(-\infty,\zeta)$ and a point mass q at $x = \zeta$. Then

$$\begin{aligned} \mu(\theta) &= E[\log u(Z)] \\ &\leq \log[E\, u(Z)] = 0 \end{aligned}$$

by Jensen's inequality and the fact that $E\, u(Z) = F(\zeta,\theta) + \bar{F}(\zeta,\theta) = 1$. The inequality is strict for a $\theta \neq \theta_o$ once we impose the identifiability condition:

$$P_{\theta_o}[f(x,\theta) \neq f(x,\theta_o) \quad , \quad x < \zeta] > 0 \quad .$$

In view of this and (3.6), the standard argument then leads to the existence of a strongly consistent sequence of roots.

Remark. The collection of regularity conditions used in course of our proofs is essentially the same as given by Halperin (1952) with the exception that the third derivative was not needed in our treatment. Also, the second derivative was not used in the consistency proof. On the other hand, we have assumed $\ddot{\rho}(x,\theta)$ to be continuous on $[\zeta-\varepsilon,\zeta+\varepsilon] \times B$. Also, we have formalized the identifiability condition which was not explicitly addressed by Halperin (1952).

4. APPLICATION TO MMLE OF MEHROTRA AND NANDA

With $\psi(x,\theta)$ and $\rho(x,\theta)$ defined in (3.1), let

$$c_n(\theta) \equiv E_\theta \dot{\rho}(Y_r,\theta) \; : \; \Omega \to R^k \tag{4.1}$$

and let $\hat{\theta}_n$ be a solution of the estimating equation

$$\ell_{1n}(\theta) \equiv \sum_{i=1}^{r} \dot{\psi}(Y_i,\theta) + (n-r)c_n(\theta) = 0 \quad . \tag{4.2}$$

Mehrotra and Nanda (1974) derived expressions for $\hat{\theta}_n$ under the normal and gamma models, and examined some exact properties including the bias and variance. The object of this section is to derive the asymptotic properties of this modified maximum likelihood estimator (MMLE) $\hat{\theta}_n$ including an expression of its limiting covariance matrix. Although the currently known applications are confined to location and scale parameters, the asymptotics can be treated for general parameters without added complication.

The asymptotic normality of $\hat{\theta}_n$ follows along the lines of Section 3 with appropriate adaptations of Theorems 1 and 2. However, an additional result in regard to the first-mean convergences of $n^{1/2}\dot{\rho}(Y_r,\theta)$ is needed.

This is stated in Lemma 2 and a proof is given in the Appendix. Also, in addition to the assumptions of Section 3, some smoothness conditions will be needed for the functions $c_n(\theta)$. Specifically, we assume that the $k \times k$ matrix of the partial derivatives $\dot{c}_n(\theta)$ converges to a limit $\nu(\theta)$ uniformly in $\theta \in B$, and that $\nu(\theta)$ is continuous at θ_o.

<u>Lemma 2</u>. If the function $\dot{f}(x) = \dot{f}(x,\theta_o) : X \to R^k$ is bounded, then $\lim n^{1/2} [c_n(\theta_o) - \dot{\rho}(\zeta,\theta_o)] = 0$.

To arrive at the asymptotic distribution of $\tilde{\theta}_n$, we first introduce some notation:

$$\begin{aligned}
\alpha &= \dot{\psi}(\zeta) - p^{-1} \textstyle\int_{-\infty}^{\zeta} \dot{\psi} f dx \\
\tau_1 &= \textstyle\int_{-\infty}^{\zeta} \dot{\psi}\, \dot{\psi}^* f dx - p^{-1} (\int_{-\infty}^{\zeta} \dot{\psi} f dx)(\int_{-\infty}^{\zeta} \dot{\psi} f dx)^* \\
\Sigma_1 &= \tau_1 + pq\alpha\alpha^* \\
J_1 &= -\textstyle\int_{-\infty}^{\zeta} \ddot{\psi} f dx - q\nu \quad , \quad \nu = \nu(\theta_o) \\
\Delta &= J_1^{-1} \Sigma_1 (J_1^{-1})^* \quad .
\end{aligned} \tag{4.3}$$

It is to be noted that the matrix ν and, a forteriori, J_1 are not necessarily symmetric.

<u>Theorem 3</u>. If $\{\tilde{\theta}_n\}$ be a consistent sequence of roots of the MML equation (4.2), and the aforementioned conditions holds, then $n^{1/2}(\tilde{\theta}_n - \theta_o) \xrightarrow{\mathcal{D}} N_k(0,\Delta)$ where Δ is given in (4.3).

<u>Outline of proof</u>. In line with the treatment of MLE in Section 3, we begin with a Taylor expansion

$$n^{-1/2} \dot{\ell}_{1n}(\theta_0) = M_n [n^{1/2} (\tilde{\theta}_n - \theta_o)] \tag{4.4}$$

where the matrix M_n corresponds to $-n^{-1} \partial \dot{\ell}_{1n}(\theta)/\partial\theta \equiv M_n(\theta)$. To establish that $n^{-1/2} \dot{\ell}_{1n}(\theta) \xrightarrow{\mathcal{D}} N_k(0,\Sigma_1)$, we refer to the rhs of (4.2), apply Theorem 1 with $g = \dot{\psi}$ and $h = 0$ in order to handle the random term, and use Lemma 2 to the non-random sequence $c_n(\theta_o)$. That the limiting mean is 0 follows from the relation

$$\int_{-\infty}^{\zeta} \dot{\psi} f dx + q\dot{\rho}(\zeta) = 0 \quad .$$

Also, the above choices of g and h in Theorem 1 lead to the covariance matrix Σ_1 defined in (4.3).

Next, consider $M_n(\theta)$ and employ Theorem 2 with $g = \ddot{\psi}$ and $h = 0$. Under the stated assumption about the uniform limit of $\dot{c}_n(\theta)$, it then follows that

$$M_n(\theta) \to J_1(\theta) \equiv -\int_{-\infty}^{\zeta} \ddot{\psi}(x,\theta) f(x) dx - q\nu(\theta)$$

w.p.1 and uniformly in $\theta \in B$. Finally, note that $J_1(\theta)$ is continuous at θ_o, and $J_1(\theta_o) = J_1$ defined in (4.3). The proof is completed by combining these results. ||

As for the existence of a strongly consistent sequence of MMLE $\tilde{\theta}_n$, a simple criterion can be provided when θ is real-valued. In this case, the equality $\dot{\ell}_{1n}(\tilde{\theta}_n) = 0$ can be viewed as a necessary condition for the maximization of a pseudo-likelihood function defined as

$$\ell_{1n}(\theta) \equiv \sum_{i=1}^{r} \psi(Y_i,\theta) + (n-r)C_n(\theta)$$

where $C_n(\theta) \equiv \int c_n(\theta) d\theta$. One can then employ Theorem 2 to deduce that

$$\begin{aligned} n^{-1}[\ell_{1n}(\theta) - \ell_{1n}(\theta_o)] \to \mu_1(\theta) \equiv \int_{-\infty}^{\zeta} \log[f(x,\theta)/f(x)] f(x) dx \\ + q[C(\theta) - C(\theta_o)] \end{aligned} \tag{4.5}$$

w.p.1 and uniformly in $\theta \in B$. Then pursuing the same lines of reasoning as for the MLE, we have

<u>Theorem 4</u>. Assume θ is real-valued and $\mu_1(\theta)$ as defined in (4.5). If $\dot{\mu}_1(\theta_o) = 0$ and $\mu_1(\theta)$ is strictly concave in a neighborhood of θ_o, then a strongly consistent sequence of MMLE exists.

A simple criterion such as Theorem 4 does not emerge in the case of a vector parameter for the reason that the construction of a pseudo-likelihood is not generally feasible. One would need to employ appropriate special methods to handle the individual problems.

5. ASYMPTOTIC EFFICIENCY RESULTS FOR LOCATION AND SCALE

The function $c_n(\theta)$ takes a simple form when θ is either a location or scale parameter. This is why the Mehrotra and Nanda MMLE has been found convenient for these cases, especially under the normal distribution. In this section we derive the asymptotic efficiency (AE) of $\tilde{\theta}_n$ relative to the MLE $\hat{\theta}_n$ for the location and scale models that are 'regular' in the sense that the conditions of the preceding sections hold. Numerical values of the AE are then computed for the normal model, and the effect of the amount of censoring is examined.

5.1 Location

Here $f(x,\theta) = f(x-\theta)$ and we take $\theta_o = 0$ without loss of generality because both $\tilde{\theta}_n$ and $\hat{\theta}_n$ are equivariant. Henceforth, we use a prime to denote differentiation wrt x while, as before, an upper dot denotes a derivative wrt θ. Defining, $\psi(x) = \log f(x)$, and the hazard rate function $\lambda(x) = f(x)/\bar{F}(x)$, we have

$$\dot{\psi}(x,\theta) = -\psi'(x-\theta),\ \dot{\rho}(x,\theta) = \lambda(x-\theta) \quad ,$$
$$c_n(\theta) = E_o\lambda(Y_r) \equiv c_n \quad . \tag{5.1}$$

Here $c_n(\theta)$ is free of θ so $\dot{c}_n(\theta) = 0$ and $C_n(\theta) = \theta c_n$. Also, $\lim c_n = \lambda(\zeta)$.

For consistency of $\tilde{\theta}_n$, we refer to (4.5) and obtain

$$\mu_1(\theta) = \int_{-\infty}^{\zeta} [\psi(x-\theta) - \psi(x)]f(x)dx + q\lambda(\zeta)\theta$$

whose derivatives at $\theta = 0$ are $\dot{\mu}_1(0) = 0$, and $\ddot{\mu}_1(0) = \int_{-\infty}^{\zeta} \psi''(x)f(x)dx$. In order that $\ddot{\mu}_1(0)$ be negative, a simple sufficient condition is that log f(x) be concave. Hence, the existence of a consistent MMLE $\tilde{\theta}_n$ is ensured by the condition that the pdf $f(x)$ is strongly unimodal.

For the asymptotic normality, we further assume that $f'(x)$ is bounded so Lemma 2 and Theorem 3 hold. Noting that $\int_{-\infty}^{\zeta}\psi'(x)f(x)dx = f(\zeta)$, and using (4.3) we obtain

$$\begin{aligned}\alpha &= -\psi'(\zeta) + p^{-1}f(\zeta)\\ \tau_1 &= \int_{-\infty}^{\zeta}[\psi'(x)]^2 f(x)dx - p^{-1}f^2(\zeta)\\ J_1 &= -\int_{-\infty}^{\zeta}\psi''(x)f(x)dx \ .\end{aligned} \tag{5.2}$$

Let $AV(\tilde{\theta}_n)$ denote the asymptotic variance of $n^{1/2}(\tilde{\theta}_n-\theta_o)$. Using the expression for Δ given in (4.3), we have

$$AV(\tilde{\theta}_n) = \frac{\int_{-\infty}^{\zeta}[\psi'(x)]^2 f(x)dx - p^{-1}f^2(\zeta) + pq[p^{-1}f(\zeta) - \psi'(\zeta)]^2}{[\int_{-\infty}^{\zeta}\psi''(x)f(x)dx]^2} \ . \tag{5.3}$$

An alternative representation of this, in terms of truncated moments, will be convenient for calculation. With X_ζ denoting a random variable which has the (truncated) pdf $p^{-1}f(x)$, $x < \zeta$, (5.3) reduces to

$$AV(\tilde{\theta}_n) = \frac{\text{Var}\ \psi'(X_\zeta) + q[E\psi'(X_\zeta) - \psi'(\zeta)]^2}{p[E\psi''(X_\zeta)]^2} \ . \tag{5.4}$$

The asymptotic variance of $n^{1/2}(\hat{\theta}_n-\theta)$, denoted by $AV(\hat{\theta}_n)$, is similarly obtained from the general expression in Section 3:

$$AV(\hat{\theta}_n) = J^{-1} = p^{-1}[\text{Var}\ \psi'(X_\zeta) + q^{-1}E^2\psi'(X_\zeta)] \ . \tag{5.5}$$

The AE of $\tilde{\theta}_n$ is then given by $AV(\hat{\theta}_n)/AV(\tilde{\theta}_n)$. These are computed for the normal distribution in Example 1.

<u>Example 1</u>. Consider $f(x) = \phi(x)$, the standard normal pdf, and let $\Phi(x)$ denote its df. Here $\psi'(x) = -x$, $\psi''(x) = -1$, and the asymptotic

variances involve the truncated moments

$$a_j = p^{-1}\int_{-\infty}^{\zeta} x^j \phi(x)dx, \ j = 1,2,\ldots,$$
$$\mu_\zeta = a_1 \ , \ \sigma_\zeta^2 = a_2 - a_1^2 \ . \tag{5.6}$$

Specifically, the AE $e_1(\zeta)$ of the Mehrotra and Nanda MMLE $\tilde{\theta}_n$ in this case is given by

$$e_1(\zeta) = [(\sigma_\zeta^2 + q^{-1}\mu_\zeta^2)(\sigma_\zeta^2 + q(\zeta - \mu_\zeta)^2)]^{-1} \ . \tag{5.7}$$

Numerical computation is simplified by the fact that both μ_ζ and σ_ζ^2 can be expressed in terms of $\phi(\zeta)$ and $\Phi(\zeta)$ by integration by parts. In particular,

$$\mu_\zeta = -\phi(\zeta)/\Phi(\zeta) \ , \ a_2 = 1 - \zeta\phi(\zeta)/\Phi(\zeta) \ ,$$
$$\sigma_\zeta^2 = 1 - \zeta\phi(\zeta)/\Phi(\zeta) - [\phi(\zeta)/\Phi(\zeta)]^2 \ . \tag{5.8}$$

Table 1 presents the values of $e_1(\zeta)$, $AV(\hat{\theta}_n)$ and $AV(\tilde{\theta}_n)$ for different ζ's and corresponding p's. Note that smaller values of p correspond to the increased severity of censoring. Although, as $p \to 0$, the limiting value of $e_1(\zeta)$ is zero, Table 1 shows an extremely slow approach to this limit. A very high efficiency is retained even for 50% censoring.

5.2 Scale

We consider $f(x,\theta) = \theta^{-1}f(x/\theta)$ and take $\theta_o = 1$ without loss of generality. With $\psi(x)$ and $\lambda(x)$ defined as before, we have

$$\psi(x,\theta) = -\log\theta + \psi(x/\theta) \ ,$$
$$\dot{\psi}(x,\theta) = -\theta^{-1} - \theta^{-2}x\psi'(x/\theta) \ , \tag{5.9}$$
$$\dot{\rho}(x,\theta) = \theta^{-2}x\lambda(x/\theta) \ .$$

Also, $c_n(\theta) = \theta^{-1}k_n$ where $k_n = E_1[Y_r\lambda(Y_r)]$ is free of θ, and $\lim k_n = \zeta\lambda(\zeta)$. Consequently, $\dot{c}_n(\theta) = -k_n\theta^{-2}$, $C_n(\theta) = k_n \log\theta$, and $\nu(\theta_o) = -\zeta\lambda(\zeta)$.

In this case, expression (4.5) becomes

$$\mu_1(\theta) = -p\log\theta + \int_{-\infty}^{\zeta} [\psi(x/\theta) - \psi(x)]f(x)dx + q\zeta\lambda(\zeta)\log\theta$$

whose derivatives at 1 are $\dot{\mu}_1(1) = 0$, and

$$\ddot{\mu}_1(1) = \int_{-\infty}^{\zeta} [x\psi'(x) + x^2\psi''(x)]f(x)dx \quad .$$

Therefore, a simple sufficient condition for the strong consistency of the MMLE $\tilde{\theta}_n$ is that $[x\psi'(x) + x^2\psi''(x)] < 0$ for all x. For the standard normal distribution, this function reduces to $-2x^2$ so the condition holds. Incidentally, it can be seen that if X is a positive random variable, this condition is equivalent to strong unimodality of the pdf of $\log X$.

The asymptotic variances $AV(\hat{\theta}_n)$ and $AV(\tilde{\theta}_n)$ again follow from the general results obtained in Sections 3 and 4 as we specialize to the functions given in (5.9). The algebra is straightforward although somewhat more tedious than the location case. Here, use of Lemma 2 requires that $f(x)$ and $xf'(x)$ be bounded. Some relevant expressions are

$$\begin{aligned}
\alpha &= -\zeta\psi'(\zeta) + p^{-1}\int_{-\infty}^{\zeta} x\psi'(x)f(x)dx \quad ,\\
\tau_1 &= \int_{-\infty}^{\zeta} [1 + x\psi'(x)]^2 f(x)dx - p^{-1}[\int_{-\infty}^{\zeta} (1 + x\psi'(x))f(x)dx]^2 \quad ,\\
J_1 &= -\int_{-\infty}^{\zeta} [1 + 2x\psi'(x) + x^2\psi''(x)]f(x)dx + q\zeta\lambda(\zeta) \quad ,\\
J &= \int_{-\infty}^{\zeta} [1 + x\psi'(x)]^2 f(x)dx + q^{-1}\zeta^2 f^2(\zeta) \quad .
\end{aligned} \tag{5.10}$$

From these basic quantities, the asymptotic variances

$$\begin{aligned}
AV(\tilde{\theta}_n) &= (\tau_1 + pq\alpha^2)/J_1^2 \\
AV(\hat{\theta}_n) &= 1/J
\end{aligned} \tag{5.11}$$

are readily obtained.

Example 2. For the normal distribution $f(x,\theta) = \theta^{-1}\phi(x/\theta)$, we have $\psi'(x) = -x$, $\psi''(x) = -1$, and the expressions (5.10) reduce in terms of the truncated moments a_j defined in (5.6). Denoting $\nu_4 = a_4 - a_2^2$, we obtain

$$\alpha = \zeta^2 - a_2 \quad , \quad \tau = p\nu_4$$

$$J_1 = 2pa_2 \quad , \quad J = p[\nu_4 + q^{-1}(a_2-1)^2] \quad .$$

Consequently,

$$AV(\tilde{\theta}_n) = p^{-1}[\nu_4 + q(\zeta^2-a_2)](2a_2)^{-2} \quad ,$$

$$AV(\hat{\theta}_n) = p^{-1}[\nu_4 + q^{-1}(a_2-1)^2]^{-1} \quad , \qquad (5.12)$$

$$e_2(\zeta) \equiv AE(\tilde{\theta}_n) = AV(\hat{\theta}_n)/AV(\tilde{\theta}_n) \quad .$$

Numerical computations are presented in Table 2. As in the case of MMLE of a normal mean studied in Example 1, the MMLE of the standard deviation does not incur much loss of AE when p is large, that is, censoring is light. Also, the AE tends to decrease in the extremely low range of p. However, it is curious that unlike the monotone behavior found in Example 1, here $AV(\tilde{\theta}_n)$ and $e_2(\zeta)$ have humps over an intermediate range of p.

6. APPLICATION TO THE MMLE OF TIKU

In the context of estimating the mean and standard deviation of a normal distribution from a type II censored sample, Tiku (1967) proposed a linear approximation of the hazard rate term that appears in the likelihood equation. Asymptotic normality and efficiency of this type of MMLE can also be obtained from the basic tools developed in Section 2. It turns out that not only for the normal model but for a general location-scale model as well, this type of MMLE is asymptotically fully efficient. To indicate why this is so, it would suffice to consider the location model $f(x,\theta) = f(x-\theta)$ in which case the likelihood equation is

$$\dot{\ell}_n(\theta) = -\sum_{i=1}^{r} \psi'(Y_i-\theta) + (n-r)\lambda(Y_r-\theta) = 0 \tag{6.1}$$

where $\lambda(x) = f(x)/\bar{F}(x)$. Tiku's MMLE results from replacing $\lambda(Y_r-\theta)$ in (6.1) by its linear approximation $h_1(Y_r-\theta) = a(Y_r-\theta) + b$ with $a = \lambda'(\zeta)$ and $b = \lambda(\zeta) - \zeta\lambda'(\zeta)$. Referring to the development in Section 3, in particular, expressions (3.2) and (3.4), note that the function $h(x) = \lambda(x)$ is now changed to $h_1(x) = ax+b$. However, with a and b specified above, we have $h(\zeta) = h_1(\zeta)$ and $h'(\zeta) = h_1'(\zeta)$ so the results in (3.5) do not change. Likewise, in place of $\beta(\zeta,\theta) = -h'(\zeta-\theta)$ in (3.6) we now have $\dot{h}_1(\zeta,\theta) = -h'(\zeta)$. However, their difference $\to 0$ as $\theta \to 0$ so we have the same $J(\theta_o)$, and hence the same asymptotic variance as for the MLE. This clarifies Tiku's (1978) heuristic reasoning that the modified likelihood equation is "asymptotically equivalent" to the original likelihood equation.

APPENDIX

<u>Proof of Lemma 2</u>. For simplicity we suppress θ_o in notation, and first consider the case where θ is real-valued. Let $f_{r,n}(\cdot)$ denote the pdf of $Y_{r,n}$, the r-th order statistic of a random sample $Z_1,\ldots,Z_n$ from $F(x)$. Since $\dot{\rho}(x) = -\dot{F}(x)/\bar{F}(x)$, we have

$$-c_n = \int_{-\infty}^{\infty} \dot{F}(x)[\bar{F}(x)]^{-1} f_{r,n}(x)dx$$

$$= [n/(n-r)] \int_{-\infty}^{\infty} \dot{F}(x) f_{r,n-1}(x)dx$$

$$= [n/(n-r)] \; E\; a(Y_{r,n-1})$$

where $a(x) \equiv \dot{F}(x)$. Since $\dot{\rho}(\zeta) = -q^{-1}a(\zeta)$ and $(n-r)/n \to q$, it would suffice to establish that

$$\lim n^{1/2}[a(\zeta) - E\; a(Y_{r,n})] = 0 \quad . \tag{A.1}$$

Let $S_n(\cdot)$ denote the empirical cdf of $Z_1,\ldots,Z_n$. Using a representation due to Bahadur (1966) we write

$$Y_{r,n} = \zeta + V_n + R_n$$

where $V_n = f^{-1}(\zeta)[p - S_n(\zeta)]$, and $R_n = O(n^{-3/4}(\log n)^{3/4})$ w.p.1 as $n \to \infty$. A Taylor expansion of $a(Y_{r,n})$ around ζ gives

$$n^{1/2}[a(Y_{r,n}) - a(\zeta) - V_n a'(\zeta)] = W_{1n} + W_{2n} \tag{A.2}$$

where

$$\begin{aligned} W_{1n} &= n^{1/2} V_n [a'(\zeta_n) - a'(\zeta)] \quad , \\ W_{2n} &= n^{1/2} a'(\zeta_n) R_n \quad , \end{aligned} \tag{A.3}$$

and ζ_n lies between ζ and $Y_{r,n}$. Since $EV_n^2 = pq/[nf^2(\zeta)]$, $\zeta_n \to \zeta$ w.p.1, and $a'(x) = \dot{f}(x)$ is assumed bounded, we have

$$E^2|W_{1n}| < f^{-2}(\zeta)pq\; E[a'(\zeta_n) - a'(\zeta)]^2 \to 0 \quad .$$

Duttweiler (1973) establishes the order of the mean-square of Bahadur's approximation. His result entails that

$$ER_n^2 = f^{-2}(\zeta)(2pq/\pi)^{1/2} n^{-3/2}[1 + o(1)]$$

and, consequently, $EW_{2n}^2 \to 0$. Thus, we have established that $E|W_{1n}| \to 0$ and $E|W_{2n}| \to 0$ which imply that the lhs of (A.2) converges to 0 in the first mean. Since $EV_n = 0$ for all n, the result (A.1) follows.

If θ is vector-valued so are c_n and $\dot{\rho}(\zeta)$, and the above argument applies to each coordinate of $n^{1/2}[c_n - \dot{\rho}(\zeta)]$. ||

Table 1. Asymptotic Efficiency of the Mehrotra and Nanda MMLE for a Normal Mean θ

ζ	p	AV($\hat{\theta}$)	AV($\tilde{\theta}$)	$e_1(\zeta)$
-2.3263	.0100	13.7515	21.0546	.6531
-1.6449	.0500	4.3320	6.0791	.7126
-1.2816	.1000	2.7845	3.7086	.7508
-.8416	.2000	1.8741	2.3395	.8011
-.5244	.3000	1.5266	1.8214	.8382
-.2533	.4000	1.3393	1.5410	.8691
.0000	.5000	1.2220	1.3634	.8963
.2533	.6000	1.1425	1.2405	.9210
.5244	.7000	1.0862	1.1509	.9438
.8416	.8000	1.0457	1.0837	.9649
1.2816	.9000	1.0172	1.0334	.9843
1.6449	.9500	1.0070	1.0141	.9930
2.3263	.9900	1.0010	1.0020	.9989

Table 2. Asymptotic Efficiency of the Mehrotra and Nanda MMLE for a Normal Standard Deviation θ

ζ	p	AV($\hat{\theta}$)	AV($\tilde{\theta}$)	$e_2(\zeta)$
-2.3263	.0100	2.3729	3.1241	.7595
-1.6449	.0500	1.3177	1.4939	.8820
-1.2816	.1000	1.1758	1.2333	.9533
-.8416	.2000	1.1469	1.1469	1.0000
-.5244	.3000	1.1364	1.1763	.9661
-.2533	.4000	1.0868	1.2250	.8872
.0000	.5000	1.0000	1.2500	.8000
.2533	.6000	.8930	1.2128	.7363
.5244	.7000	.7823	1.0872	.7195
.8416	.8000	.6779	.8854	.7657
1.2816	.9000	.5843	.6699	.8722
1.6449	.9500	.5414	.5787	.9355
2.3263	.9900	.5084	.5158	.9857

REFERENCES

BAHADUR, R. R. (1966), "A Note on Quantiles in Large Samples," Annals of Mathematical Statistics, 37, 577-580.

DUTTWEILER, D. L. (1973), "The Mean-Square Error of Bahadur's Order-Statistics Approximation," Annals of Statistics, 1, 446-453.

FLIGNER, M. A., and HETTMANSPERGER, T. P. (1979), "On the Use of Conditional Asymptotic Normality," Journal of the Royal Statistical Society, Ser. B, 41, 178-183.

HALPERIN, M. (1952), "Maximum Likelihood Estimation in Truncated Samples," Annals of Mathematical Statistics, 23, 226-238.

JENNRICH, R. I. (1969), "Asymptotic Properties of Nonlinear Least Squares Estimators," Annals of Mathematical Statistics, 40, 633-643.

MEHROTRA, K. G. and NANDA, P. (1974), "Unbiased Estimation of Parameters by Order Statistics in the Case of Censored Samples," Biometrika, 61, 601-606.

SETHURAMAN, J. (1961), "Some Limit Theorems for Joint Distributions," Sankhya, Ser. A, 23, 379-386.

TIKU, M. L. (1967), "Estimating the Mean and Standard Deviation From a Censored Normal Sample," Biometrika, 54, 155-165.

__________ (1978), "Linear Regression Model With Censored Observations," Communications in Statistics, A7, 1219-1232.

GKB/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
#2616

2. GOVT ACCESSION NO.
AD A139958

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
On the Asymptotics of Maximum Likelihood and Related Estimators Based on Type II Censored Data

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Gouri K. Bhattacharyya

8. CONTRACT OR GRANT NUMBER(s)
N00014-78-C-0722
DAAG29-80-C-0041

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of
610 Walnut Street Wisconsin
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Work Unit Number 4 - Statistics and Probability

11. CONTROLLING OFFICE NAME AND ADDRESS
See Item 18 below

12. REPORT DATE
December 1983

13. NUMBER OF PAGES
23

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

Office of Naval Research
Department of the Navy
Arlington, VA 22217

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Censored data; Maximum likelihood; Modified maximum likelihood; Asymptotic theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Some simple procedures are provided for establishing the asymptotic normality and uniform strong convergence of a class of functions that arise in the context of estimating parameters from a type II censored sample. These are used to streamline and strengthen the traditional treatment of the asymptotic theory of maximum likelihood estimators based on censored data. Further applications include the treatment of asymptotics of some modified maximum likelihood (MML) estimators. In particular, conditions are provided for the consistency and limiting normality of the MML estimators of Mehrotra and Nanda, and the asymptotic efficiencies of these estimators are evaluated.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

FILM

3-8